QP CODE: 26022393

26022393

Reg No : .....

Name : .....

M Sc DEGREE (CSS) EXAMINATION, MAY 2026

Second Semester

M Sc MATHEMATICS

CORE COURSE - ME010201 - ADVANCED ABSTRACT ALGEBRA

2019 ADMISSION ONWARDS

6784D905

Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Find the degree and a basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$
2. *Define an algebraically closed field. Check $\mathbb{Q}$ is algebraically closed.*
3. Define primitive polynomial with example.
4. Define a Euclidean domain. Give an example.
5. Prove that $\mathbb{Z}[i]$ is an integral domain.
6. State the Conjugation Isomorphism theorem.
7. *State the isomorphism extension theorem. Use this theorem to prove that if $E \leq \overline{F}$ is an algebraic extension of a field F and $\alpha, \beta \in E$ are conjugate over F, then the conjugation isomorphism $\psi_{\alpha,\beta} : F(\alpha) \to F(\beta)$ can be extended to an isomorphism of E onto a subfield of $\overline{F}$.*
8. *Prove that the splitting field over $\mathbb{Q}$ of $x^3 - 2$ is of degree 6 over $\mathbb{Q}$.*
9. *Describe the group of the polynomial $x^4 - 5x^2 + 6 \in \mathbb{Q}[x]$ over $\mathbb{Q}$.*
10. *Define the nth cyclotomic extension of a field F. Give an example.*

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. a) Show that the polynomial x^2+1 is irreducible in $\mathbb{Z}_3[x]$.

    b) Let α be a zero of x^2+1 in an extension field of $\mathbb{Z}_3$ .Give the addition and multiplication tables for the nine elements of $\mathbb{Z}_3(\alpha)$.
12. Prove that the set of all constructible real numbers forms a subfield of the field of real numbers.

13. Define PID and UFD. Show with an example that not every UFD is a PID.

14. If D is an integral domain with a multiplicative norm N, then prove that N(1) = 1 and |N(u)| = 1 for every unit u in D. If furthermore, every α such that |N(α)| = 1 is a unit in D, then prove that an element π in D, with |N(π)| = p for a prime p in Z , is an irreducible of D.

15. *Let* $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ *and* $F = \mathbb{Q}$. *Let* $\sigma_1 = \psi_{\sqrt{2},-\sqrt{2}}$, $\sigma_2 = \psi_{\sqrt{3},-\sqrt{3}}$ *and* $\sigma_3 = \sigma_1\sigma_2$. *Find the fixed fields* $E_{\{\sigma_1,\sigma_3\}}$, $E_{\{\sigma_3\}}$ *and* $E_{\{\sigma_2,\sigma_3\}}$.

16. *If* $F \leq E \leq K$, *where* K *is a finite extension field of a field* F, *then prove that* $\{K : F\} = \{K : E\}\{E : F\}$. *Illustrate this result with an example.*

17. Let $\overline{F}$ be an algebraic closure of a field F and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a monic polynomial in $\overline{F}[x]$. *If* $(f(x))^m \in F[x]$ *and* $m \cdot 1 \neq 0$ *in* F, *prove that* $f(x) \in F[x]$, *that is, all* $a_i \in F$.

18. Let K be a finite normal extension of a field F and let E be an extension of F, where $F \leq E \leq K \leq \overline{F}$. Prove the following.
    a) K is a finite normal extension of E.
    b) $G(K/E)$ is precisely the subgroup of $G(K/F)$ consisting of all those automorphisms that leave E fixed.
    c) Two automorphisms σ *and* τ *in* $G(K/F)$ *induce the same isomorphism of* E *onto a subfield of* $\overline{F}$ *if and only if they are in the same left coset of* $G(K/E)$ *in* $G(K/F)$.

(6×2=12 weightage)

Part C (Essay Type Questions)

Answer any **two** *questions.*

Weight **5** *each.*

19. a) Prove that a finite field $GF(p^n)$ of p^n elements exists for every prime power p^n
    b) If F *is a finite field, then prove that for every positive integer* n *there is an irreducible polynomial in* $F[x]$ *of degree* n.

20. a) Let D be a PID. Prove that every element that is neither 0 nor a unit in D is a product of irreducibles.
    b) Prove that an ideal (p) in a PID is maximal if and only if p is an irreducible.

21. *Define splitting field over a field* F. *Prove that a field* E, $F \leq E \leq \overline{F}$, *is a splitting field over* F *if and only if every automorphism of* $\overline{F}$ *leaving* F *fixed maps* E *onto itself.*

22. Prove the following.
    a) Let F be a field and $f(x)$ be an irreducible polynomial in $F[x]$. Then all zeros of $f(x)$ in $\overline{F}$ have the same multiplicity.
    b) Let F be a field and $f(x)$ be an irreducible polynomial in $F[x]$. Then $f(x)$ has a factorization in $\overline{F}[x]$ of the form $a\prod_i(x - \alpha_i)^{\nu}$ where α_i are the distinct zeros of $f(x)$ in $\overline{F}$ and $a \in F$.
    c) If E is a finite extension of a field F, then $\{E : F\}$ divides $[E : F]$.

(2×5=10 weightage)